

# From compositional to systematic semantics

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## Abstract

We prove a theorem stating that any semantics can be encoded as a compositional semantics, which means that, essentially, the standard definition of compositionality is formally vacuous. We then show that when compositional semantics is required to be "systematic" (that is, the meaning function cannot be arbitrary, but must belong to some class), it is possible to distinguish between compositional and non-compositional semantics. As a result, we believe that the paper clarifies the concept of compositionality and opens a possibility of making systematic formal comparisons of different systems of grammars.

## 1 Introduction

Compositionality is defined as the property that the meaning of a whole is a function of the meaning of its parts (cf. e.g. [Keenan and Faltz1985], pp.24-25). (A slightly less general definition, e.g. [Partee *et al.*1990], postulates the existence of a homomorphism from syntax to semantics). However, we can prove a theorem stating that any semantics can be encoded as a compositional semantics, which means that, essentially, the standard definition of compositionality is formally vacuous. Thus, although intuitively clear, the definition is not restrictive enough. We illustrate the power of the theorem by showing how to assign compositional semantics to idioms and to a very counterintuitive semantics of coordination (Section 4).

Given a class of functions  $F$ , we say that the compositional semantics is *systematic* if the meaning function belongs to the class  $F$ . We show that when compositional semantics is required to be *systematic* we can distinguish between grammars with compositional and non-compositional semantics; we present an example of a simple grammar for which there is no "systematic" compositional semantics (Section 3).

As a result, we believe that the paper clarifies the concept of compositionality and opens a possibility of making systematic comparisons of different systems of grammars and natural language understanding programs. Furthermore, the concept of systematicity that we introduce in the paper might be useful in extracting the formal meaning behind various versions of compositionality as a philosophical principle, cf.[Partee1982]. But in this paper we restrict ourselves to the mathematics of compositionality.

Compositional semantics is usually defined as a functional dependence of the meaning of an expression on the meanings of its parts. One of the first natural questions we might want to ask is whether a set of natural language expressions, i.e. a language, can have some compositional

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semantics. This question has been answered positively by [van Benthem1982]. However his result says nothing about what kinds of things should be assigned e.g. to nouns, where, obviously, we would like nouns to be mapped into sets of entities, or something like that. That is, we want semantics to encode some basic intuitions, e.g. that nouns denote sets of entities, and verbs denote relations between entities, and so on; in other words, we would like to have a compositional semantics that agrees with intuitions. More formally, the question is whether after deciding what sentences and their parts mean, we can find a function that would compose the meaning of a whole from the meanings of its parts.

The answer to this question is somewhat disturbing. It turns out that whatever we decide that some language expressions should mean, it is always possible to produce a function that would give compositional semantics to it (see below for a more precise formulation of this fact). The upshot is that compositionality, as commonly defined, is not a strong constraint on a semantic theory.

## 2 Proving the existence of compositional semantics

Let  $S$  be any collection of expressions (intuitively, sentences and their parts). Assume that elements of  $S$  (e.g.  $s.t$ ) are composed from other elements of  $S$  (that is,  $s$  and  $t$ ) by concatenation ("."). We do not assume that concatenation is associative, that is  $(a.(b.c)) = ((a.b).c)$ . Intuitively, this means that we assign semantics to parse trees, and not to strings of words.

Let  $M$  be a set of meanings, and let for any  $s \in S$ ,  $m(s) \in M$  be the meaning of  $s$ . We want to show that there is a compositional semantics for  $S$  which agrees with the function  $m$  associating  $s$  with  $m(s)$ .

Since elements of  $M$  can be of any type, we do not automatically have  $m(s.t) = m(s) \sharp m(t)$ , where  $\sharp$  is some operation on the meanings. To get that kind of homomorphism we have to perform a type raising operation that would map elements of  $S$  into functions and then the functions into the required meanings. Note that such a type raising operation is quite common both in mathematics (e.g. 1 being a function equal to 1 for all values) and in mathematical linguistics. The meaning function  $\mu$  that we want to define will provide compositional semantics for  $S$  by mapping it into a set of functions in such a way that  $\mu(s.t) = \mu(s)(\mu(t))$ , for all elements  $s.t$  of  $S$ .

Secondly, we want that the original semantics be easily decoded from  $\mu(s)$ . There is more than one way of doing this. One can trivially extend the language  $S$  by adding to it an "end of expression" character  $\$$ , which may appear only as the last element of any expression. The purpose of it is to encode the function  $m(x)$  in the following way:  $\mu(s.\$) = m(s)$ , for all  $s$  in  $S$ . Intuitively, the character  $\$$  is like the period at the end of a sentence, or the pause marking the end of an utterance. In effect, we will be treating all sentences as idioms, or garden path sentences, where the meanings are clear only once the sentence is completed (Theorem 2). But, as we are going to show now, the original semantics can be encoded in a different way, without extending the original language, e.g. by assuring  $\mu(s)(s) = m(s)$ , for all  $s$  in  $S$  (Theorem 1).

To make the notation simple, we have assumed that there is only one way of composing elements of  $S$ , by concatenation, but all our arguments work for languages with many operators as well. We show an example of how such operators can be handled in Section 4.

**THEOREM 1.** Let  $M$  be an arbitrary set. Let  $A$  be an arbitrary alphabet. Let "." be a binary operation, and let  $S$  be the set closure of  $A$  under ".". Let  $m : S \rightarrow M$  be an arbitrary function.

Then there is a set of functions  $M^*$  and a unique map  $\mu : S \rightarrow M^*$  such that for all  $s, t \in S$

$$\mu(s.t) = \mu(s)(\mu(t)), \quad \text{and} \quad \mu(s)(s) = m(s).$$

**Corollary 2.** Theorem 1 is also true when the binary operation "." is partial.

### Preliminaries to the proof: The solution lemma

Our results will be proved in set theory with the anti-foundation axiom. This set theory, ZFA, is equiconsistent with the standard system of ZFC, thus the theorem does not assume anything more than what is needed for "standard mathematical practice". Furthermore, ZFA is better suited as foundations for semantics of natural language than ZFC ([Barwise and Etchemendy1987]).

We need only one (but fundamental) theorem of ZFA: the solution lemma ([Aczel1987] and [Barwise and Etchemendy1987]), which says any (well-formed) collection of equations that define sets has a unique solution. For the reader who is not familiar with set theory, the meaning of the solution lemma can be explained as follows: We have a universe of sets  $V$ , and a set of variables  $X = \{x_1, x_2, \dots\}$ , which may be infinite (countable or uncountable). We can form equations of the form

$$x_i = a\_set(X, V)$$

where  $a\_set(X, V)$  is a set expression involving the variables and elements of  $V$ , for instance, if  $a \in V$  and  $b \in V$ , we can write the following equations:

$$\begin{aligned} x_7 &= \{\{a\}, x_7, x_9\} \\ x_8 &= \{b, \{a, x_8\}, \{\{\{x_{79}\}\}\}\} \\ x_9 &= \{b\} \\ x_{79} &= \{x_{79}\} \end{aligned}$$

We say that such a set is well-formed if each variable appear only once on the left, and each left hand side is a variable. The solution lemma says that any set of such equations (finite or infinite) has a unique solution. That is, there is a unique collection of sets that satisfy them.  $\square$

### PROOF OF THEOREM 1 AND COROLLARY 2

**Proof of Theorem 1.** It is enough to ensure that for all  $s \in S$

$$\mu(s) = \{< s, m(s) >\} \cup \{< \mu(t), \mu(s.t) > : t \in S\}$$

Clearly,  $\mu$  is a function, because it is a collection of pairs. The proof is complete once we check that for  $s$ 's and  $t$ 's in  $S$  we have (i)  $\mu(s.t) = \mu(s)(\mu(t))$ , and (ii)  $\mu(s)(s) = m(s)$ . Using the above equality we check (i): If  $f = \mu(s)$ , then  $f(\mu(t)) = \mu(s.t)$ . Similarly for (ii).

It remains to show that using the solution lemma we can make the above equation true for all  $s \in S$ . We begin by introducing a set variable  $X_s$  for every  $s \in S$ , and observing that

$$X_s = \{< s, m(s) >\} \cup \{< X_t, X_{s.t} > : t \in S\}$$

is a well-formed set equation for any  $s \in S$ . (The pair  $< a, b >$  is set theoretically defined as  $\{\{a\}, \{a, b\}\}$ ). Hence the solution lemma applies, and there are unique sets  $\mu(s)$  that satisfy each

equation. But each such  $\mu(s)$  is a collection of pairs, i.e. a function. Furthermore, since each  $\mu(s)$  is unique, and  $S$  is a set, the mapping  $\mu$  associating  $\mu(s)$  with each  $s \in S$  is a function. This completes the proof of Theorem 1. □

**Proof of Corollary 2.** It is enough to observe that we can add an extra condition in the main equation of Theorem 1, and the proof still works:

$$\mu(s) = \{ \langle s, m(s) \rangle \} \cup \{ \langle \mu(t), \mu(s.t) \rangle : t \in S \text{ and } s.t \in S \}$$

□

NOTE. Notice that we can view using the solution lemma in the above proofs as an extreme example of defining a function by cases. To see it more clearly, one can make the main equation of the proof of Theorem 1 explicit. Let  $t(0), t(1), \dots, t(\alpha)$  enumerate  $S$ . We can create a big table specifying meaning values for all strings and their combinations. Then the conditions on the meaning functions  $\mu(s)$  can be written as the set of equations below

$$\begin{aligned} \mu(t(0)) &= \{ \langle t(0), m(t(0)) \rangle, \langle \mu(t(0)), \mu(t(0).t(0)) \rangle, \dots, \\ &\quad \langle \mu(t(\alpha)), \mu(t(0).t(\alpha)) \rangle, \dots \} \\ \mu(t(1)) &= \{ \langle t(1), m(t(1)) \rangle, \langle \mu(t(0)), \mu(t(1).t(0)) \rangle, \dots, \\ &\quad \langle \mu(t(\alpha)), \mu(t(1).t(\alpha)) \rangle, \dots \} \\ &\dots \\ \mu(t(\alpha)) &= \{ \langle t(\alpha), m(t(\alpha)) \rangle, \langle \mu(t(0)), \mu(t(\alpha).t(0)) \rangle, \dots, \\ &\quad \langle \mu(t(\alpha)), \mu(t(\alpha).t(\alpha)) \rangle, \dots \} \\ &\dots \end{aligned}$$

In ordinary mathematics, this would correspond to saying that if  $x$  is 1 then  $f(x) = 32$ , if  $x$  is 2 then  $f(x) = 14732$ , if  $x$  is 3 then  $f(x) = 1$ , and so on. Clearly, such a process defines the function  $f$ , but, intuitively, it is not a definition we would care much for. Before showing that requiring a better description of an  $f$  than as a set of pairs makes sense, we want to observe that the encoding of the original meaning function can be uniform in the following sense:

**PROPOSITION 3.** In addition to the assumptions of Theorem 1, let  $\$ \notin A$ , and let  $S\$$  be the language obtained by the mapping  $s \rightarrow s.\$,$  for all  $s \in S$ . Then there is a set of functions  $M^*$  and a unique map  $\mu : S\$ \rightarrow M^*$  such that for all  $s, t \in S$

$$\mu(s.t) = \mu(s)(\mu(t)), \text{ and } \mu(s.\$) = m(s).$$

Proof. As in the proof of Corollary 2, we can change the set of equations to

$$\mu(s) = \{ \langle \$, m(s) \rangle \} \cup \{ \langle \mu(t), \mu(s.t) \rangle : t \in S \}$$

To finish the construction of  $\mu$ , we make sure that the equation  $\mu(\$) = \$$  holds. Formally, this requires adding the pair  $\langle \$, \$ \rangle$  into the graph of  $\mu$  that was obtained from the solution lemma.

Also, we have to extend the domain of function  $\mu$  to include  $S\$$ . This is easily done by adding to the already constructed part of  $\mu$  the set of pairs  $\{ \langle s\$, m(s) \rangle : s \in S \}$ . The proof is complete once we check that for  $s$ 's in  $S$  we have  $\mu(s\$) = m(s)$ , and that  $\mu(s\$) = \mu(s)(\mu(\$))$  (because  $\mu(\$) = \$$ , and, according to the equation,  $\mu(s)(\$) = m(s)$ ).

□

Note that, as in Corollary 2, if a certain string does not belong to the language, we can assume that the corresponding value in this table is undefined; thus  $\mu$  is not necessarily defined for all possible concatenations of strings of  $S$ .

In view of the above theorems, any semantics is equivalent to a compositional semantics, and hence it would be meaningless to keep the definition of compositionality as the existence of a homomorphism from syntax to semantics without imposing some conditions on this homomorphism. Notice that requiring the computability of the meaning function will not do. In mathematics, where semantics obviously is compositional, we can talk about noncomputable functions, and it is usually clear what we postulate about them. Also, we have the following proposition.

**PROPOSITION 4.** If the set of expressions  $S$  and the original meaning function  $m(x)$  are computable, then so is the meaning function  $\mu(x)$ .

Proof. One can easily check that the table defining the meaning functions  $\mu(t(\alpha))$  in the note above is effectively computable from the functions  $m(x)$  and  $t(\alpha)$ . Hence so is the function  $\mu$ . □

NOTE. What does it mean that the table is effectively computable from the functions  $m(x)$  and  $t(\alpha)$ ? It means that given a Turing machine,  $T1$ , that prints all elements of  $S$ , and another Turing machine,  $T2$ , that takes an element  $s$  on the output tape of  $T1$  as input and produces as output  $m(s)$ , we can construct a third Turing machine,  $T3$ , that produces the successive elements of the table, i.e. enumerates all the equations (e.g. for any pair  $\langle m, n \rangle$  gives the  $n$ th value pair of the  $m$ th equation). But these equations define our function  $\mu$ . Hence  $\mu$  is effectively computable. Also, notice that the proposition holds true for generalized computability, in the sense of [Barwise1975].

### 3 Systematic semantics. I. Examples

[Hirst1987], pp.27, talks about compositionality, postulating that the meaning of a whole should be a "systematic" function of the meanings of the parts. He does not define the word "systematic" except as being an antonym to "idiosyncratic". What it could mean is that we want to avoid such meaning functions as the ones defined in the proof of Theorem 1 and the subsequent propositions. We suggest a simple way of doing it — by requiring that the meaning function belong to a certain class. By Proposition 4 this does not work if we merely postulate that the function be computable. However it does work for smaller classes of functions as the following examples show.

#### A simple grammar without a systematic semantics

If meanings have to be expressed using certain natural, but restricted, sets of operations, it may turn out that even simple grammars do not have a compositional semantics. Consider two grammars of numerals in base 10:

**Grammar ND** $N \leftarrow N D$  $N \leftarrow D$  $D \leftarrow 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9$ **Grammar DN** $N \leftarrow D N$  $N \leftarrow D$  $D \leftarrow 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9$ 

**PROPOSITION 5.** For the grammar ND, the meaning of any numeral can be expressed in the model  $(Nat, +, *, 10)$  as

$$\mu(ND) = 10 * \mu(N) + \mu(D)$$

that is, a polynomial in two variables with coefficients in natural numbers. □

On the other hand, for the grammar DN, we can prove that no such a polynomial exists:

**THEOREM 6.** There is no polynomial  $p$  in two variables  $x, y$  such that

$$\mu(DN) = p(\mu(D), \mu(N))$$

and such that the value of  $\mu(DN)$  is the number expressed by the string  $DN$  in base 10.

Proof. We are looking for

$$\begin{aligned} \mu(DN) &= p(\mu(D), \mu(N)) \\ &= \mu(D) * 10^{\text{length}(N)} + \mu(N) \end{aligned}$$

where the function  $p$  must be a polynomial in these two variables. If such a polynomial exists, it would have to be equal to  $\mu(N)$  for  $\mu(N)$  in the interval  $0..9$ , and to  $\mu(D) * 10 + \mu(N)$  for  $\mu(N)$  in  $10..99$ , and to  $\mu(D) * 100 + \mu(N)$  for  $\mu(N)$  in  $100..999$ , and so on. Let the degree of this polynomial be less than  $n$ , for some  $n$ . Let us consider the interval  $10^n..10^{(n+1)} - 1$ . On this interval the polynomial would have to be equal identically to  $\mu(D) * 10^n + \mu(N)$ . Now, if two polynomials of degrees less than  $n$  agree on  $n$  different values, they must be identical. Hence,  $p(\mu(N), \mu(D)) = \mu(D) * 10^n + \mu(N)$ . But this would give wrong values for other intervals, e.g.  $10..99$ . Contradiction. □

But notice that there is a compositional semantics for the grammar DN that does not agree with intuitions:  $\mu(DN) = 10 * \mu(N) + \mu(D)$ , which corresponds to reading the number backwards. And there are many other semantics corresponding to all possible polynomials in  $\mu(D)$  and  $\mu(N)$ . Also observe that (a) if we specify enough values of the meaning function we can exclude any particular polynomial; (b) if we do not restrict the degree of the polynomial, we can write one that would give any values we want on a finite number of words in the grammar. The moral is that not only it is natural to restrict meaning functions to, say, polynomials, but to further restrict them. E.g. if

we restrict the meaning functions to polynomials of degree 1, then by specifying only three values of the meaning function we can (a) have a unique compositional semantics for the first grammar; and (b) show that there is no compositional semantics for the second grammar (directly from the proof of the above theorem).

## 4 Some linguistic examples

In this section we want to explain how the theorems we proved in Section 2 apply to typical linguistic examples. In the process we also explain how languages with operators can be handled by our method. We begin by discussing the simple case of idioms.

**Idioms.** Intuitively, the meaning of "high seas" is not compositional, because "high" refers to length or distance, and not to open spaces; moreover one could even argue that although "seas" is plural, "high seas" is semantically singular, for it means an "open sea". (However, the precise semantics of the expression is not important at this point). We want to show how we can assign compositional semantics to such non-compositional examples.

Let the language  $S$  consist of *wall*, *seas*, *high*, *high.wall*, and *high.seas*. The equations we need to ensure the compositionality of semantics have the familiar form:

$$\begin{aligned}\mu(seas) &= \{ \langle seas, m(seas) \rangle \} \\ \mu(wall) &= \{ \langle wall, m(wall) \rangle \} \\ \mu(high) &= \{ \langle high, m(high) \rangle, \langle \mu(wall), \mu(high.wall) \rangle, \\ &\quad \langle \mu(seas), \mu(high.seas) \rangle \} \\ \mu(high.seas) &= \{ \langle (high.seas), open(m(seas)) \rangle \} \\ \mu(high.wall) &= \{ \langle (high.wall), high(m(wall)) \rangle \}\end{aligned}$$

Notice, that we could add *building* and other words to the language and easily extend this set of equations. The intuition we associate with compositionality would be captured by the uniformity of the meanings of *high.X* as *high(m(X))*, where  $X$  ranges over *wall*, *building*, .... However the formal expression of this intuition as the principle of compositionality does not work, which can be seen by noticing that the meaning of *high.seas* is a composition of the meaning functions for *high* and *seas*. What is happening has to do with the fact that, by definition, functions defined by cases are as good as any others. And what we have done is to have defined the meaning of *high* by cases.

**Coordination.** We now turn our attention to a slightly more complicated example. Consider disjunction and conjunction,  $+$  and  $\&$ . We plan to prove the following results:

**PROPOSITION 7.** Let  $+$  and  $\&$  denote "or" and "and". Then:

- (A). It is possible to assign compositionally the "natural" semantics of  $(a + b)\&c$  to expressions of type  $a + (b\&c)$  and preserve the original meanings of  $a + b$  and  $b\&c$ .
- (B). (A) is not possible if the meaning functions have to be Boolean polynomials.

Proof. To keep things as simple as possible, consider language  $S$  consisting of  $a + (b\&c)$ ,  $b\&c$ ,  $b$ ,  $c$ , and  $a$ . To apply directly the solution lemma we should represent the operators in their prefix form; e.g.  $a + b$  becomes  $+.a.b$ . Then our language  $S$  becomes  $+.a.\&.b.c$ ,  $\&.b.c$ ,  $+$ ,  $\&$ ,  $b$ ,  $c$ , and  $a$ . (And

for the sake of completeness we can add to it  $a \& b.c$ , and  $b.c$ ). As before we write our equations (this time using the version with  $\$$ ):

$$\begin{aligned}
\mu(+.a \& b.c) &= \{ \langle \$, (m(a) + m(b)) \& m(c) \rangle \} \\
\mu(a \& b.c) &= \{ \langle \$, \langle a, m(b) \& m(c) \rangle \rangle \} \\
\mu(\& b.c) &= \{ \langle \$, m(b) \& m(c) \rangle \} \\
\mu(b.c) &= \{ \langle \$, \langle m(b), m(c) \rangle \rangle \} \\
\mu(b) &= \{ \langle \$, m(b) \rangle, \langle \mu(c), \mu(b.c) \rangle \} \\
\mu(\&) &= \{ \langle \$, m(\&) \rangle, \langle \mu(b.c), \mu(\& b.c) \rangle \} \\
\mu(a) &= \{ \langle \$, m(a) \rangle, \langle \mu(\& b.c), \mu(a \& b.c) \rangle \} \\
\mu(+) &= \{ \langle \$, m(+ ) \rangle, \langle \mu(a \& b.c), \mu(+.a \& b.c) \rangle, \\
&\quad \langle \mu(a.b), m(+.a.b) \rangle \}
\end{aligned}$$

It can be easily checked that as before  $\mu(s.t) = \mu(s)(\mu(t))$  for all  $s, t \in S$ . The meaning  $m(a)$  of  $a$  is arbitrary, but we would typically identify it with its logical value (true or false). Also, notice that without loss of generality those  $a, b, c$  (and hence the language  $S$ ) can be "expanded" to sets of variables  $a_i, b_j, c_k$ , resulting in a slight change in the equations, but not changing the content of the theorem. Then, for all pairs of variables disjunction and conjunction would behave as usual; however for a combination of a variable with a Boolean formula they could behave arbitrarily. This proves the first part.

We now prove the second part, i.e. that if the meaning functions are restricted to Boolean polynomials, it is impossible to assign compositional semantics given by the "natural" semantics of  $(a+b) \& c$  to expressions of the type  $a + (b \& c)$  and preserve the original meanings of the connectives. (The "natural" semantics is of course the conjunction of the Boolean value of  $a+b$  with the Boolean value of  $c$ ).

The proof consists in observing that  $(m(a) + m(b)) \& m(c)$  cannot be obtained as a Boolean polynomial (function)  $p(m(a), m(b) \& m(c))$ . To see it, it is enough to construct a truth table showing the values of  $(a+b) \& c$  and  $b \& c$  and observing that there cannot be a functional dependence of the former on the latter and the value of  $a$  (compare the values for the triples  $\langle a = 1, b = 0, c = 1 \rangle$  and  $\langle a = 1, b = 1, c = 0 \rangle$ ).  $\square$

## 5 Systematic semantics. II. Discussion

Above, we have argued that the existence of a homomorphism from syntax to semantics does not restrict the grammar, but if we put some constraints on such a homomorphism, they actually might restrict grammars of languages. We have called such homomorphisms *(F-)systematic*. However the nature of systematicity seems to be very much an open problem. In this section we discuss some of the most obvious issues, and propose some research possibilities in this area.

The first natural question that arises is: What should be this class  $F$ ? We have shown that for a grammar of numbers, and a grammar of two Boolean connectives the natural classes are polynomials. Clearly, this cannot always be the case. For instance, it seems natural to map verbs into predicates and nouns into their arguments. But we know that if we want to provide



compositional semantics for more than the simplest case of subject-verb-object construction we need other mechanism, e.g. type raising. On the other hand, unrestricted type raising leads to the results we have just discussed. We arrive then at the following variant of the natural question: How should we restrict type raising? (So that we can account e.g. for ellipsis, but at the same time constrain the grammars).

Many grammatical constructions express meanings that go beyond expressing predicate-arguments assignments. For example, "the X-er, the Y-er" construction ([Fillmore *et al.*1988], [Zadrozny and Manaster-Ramer1994] as in "the more you dive, the better you swim", expresses a proportional dependence. Other constructions can express speech acts, various kind of conflict, etc., hence creating rich sets of meanings. The next question we can ask is whether we can express constraints on the syntax-semantics interface by saying that the homomorphism should be simple, relative to the the class of meaning functions. A mathematical analogy would be to say that we are given many different functions, e.g.  $\sin(x)$ ,  $\cos(x)$ ,  $e^x$ , ..., but only simple ways of composing them, e.g. only by the  $+$ .

In essence, when we say that a function is *F-systematic*, we view  $F$  as a measure of complexity and/or expressive power. Hence the natural association with polynomials. But there are other measures of complexity. [Savitch1993] discusses grammars and languages in terms of Kolmogoroff complexity, suggesting that more compact grammars are better even if they overgeneralize (e.g. by approximating a finite language by an infinite one). We believe that his work is relevant for systematicity, but we do not have any formal argument supporting that claim, except the observation that polynomials are more compact than functions defined by cases. So perhaps this might be a beginning of a formal connection.

One of the referees has pointed out two other ideas. First, there could be other more natural notions of systematicity, in cases when meanings are specified by means of constraint solving, as is implicit in unification-based formalisms, and even in Theorem 1, where meanings are extracted as solutions to equations. The second idea, the differences between natural and formal languages notwithstanding, is a programming language semantics concept which actually restricts the ability of a semantics to be compositional. This concept is "full abstraction", i.e. the equivalence between the operational and denotational semantics, (cf. [Gunter1993]), which can be viewed as a general constraint on compositionality.

A radically different approach to the interaction of syntax and semantics has been presented in [Zadrozny and Manaster-Ramer1994] and [Jurafsky1992]. Language is modeled there as a set of constructions (cf. also [Fillmore *et al.*1988] and [Bloomfield1933]). In that model there is no separate syntax, since constructions encode both form and meaning. Each construction explicitly defines the meaning function taking the meanings of its subconstructions as arguments. Intuitively, in that model we assume that each word sense requires a separate semantic description; the same is true about each idiom, open idiom ([Fillmore *et al.*1988]), and a phrasal construction. This means that we make each semantic function as complicated as linguistically necessary, but their mode of combination is restricted. (Continuing the above mathematical analogy, we would say that the only mode of combination is substitution for an argument). In the construction-based model semantics is "compositional" and "systematic" (with respect to the set of all these meaning functions), but there is no homomorphism from syntax to semantics, because there is no syntax to speak of. It is "compositional", because the meaning of a construction is a function of the meanings of its parts and their mode of combination. (Note that such a function is different for different constructions, and each construction defines its own mode of combination). And it is systematic, because the modes of combination are not arbitrary, as they have to be linguistically justified. But since only few formal aspects of constructions have been worked out, we can only speak of that model as of yet another possibility.

The last point we want to make is that while it is true that the homomorphism condition is too weak (in general) to count as systematicity, the semanticists (e.g. Montague) have not been using arbitrary homomorphisms. Thus a careful examination of their work should lead to some characterization of "good" homomorphisms; and this seems to be another interesting avenue of research. (As suggested by one of the referees, a technique from universal algebra might also prove helpful, where one first gives a class of algebras and then specifies meanings as homomorphisms from the initial algebra of the class).

## 6 Conclusions

In this paper we have shown the formal vacuity of the compositionality principle. That is, we have shown that the property that the meaning of the whole is a function of the meanings of its parts does not put any material constraints on syntax or semantics. Theorem 1 (and its corollaries) explain formally why the postulate of a homomorphism between syntax and semantics is not restrictive enough: the syntactic operator of concatenation "." can be mapped into functional composition operating on functions that encode arbitrary semantics of an arbitrary language. As we have seen in the examples, the presence of other operators does not change the result, because they can be treated as yet another letter of the alphabet, and one can still produce a homomorphism between syntax and semantics.

The problem of the vacuity of compositional semantics arises, because in the formal definition of compositionality meaning functions can be completely arbitrary. Therefore we have proposed that the meaning functions should be systematic, i.e., non-arbitrary. We have shown that this notion makes sense formally; that is, we presented examples of semantic classes of functions, for which there are grammars with meaning functions in that class, as well as we have shown that there are grammars that cannot have a meaning function in that class. As we noted in the previous section, both the formal and the linguistic nature of systematicity remains an open problem, but with a few promising avenues of research.

In this paper we have restricted ourselves to the mathematics of compositionality, and many important issues have not been discussed. For instance, the main result is relevant for theories of grammar and for the thesis about the reduction of syntax to lexical meanings (cf. e.g. T. Wasow on pp.204-205 in [Sells1985]). Also, systematicity of semantics should give us a handle in constraining the power of the semantic as well as the syntactic components of a grammar (cf. [Manaster-Ramer and Zadrozny1994]). Furthermore, our results have implications for computational linguistics (they are briefly discussed in [Zadrozny1992]).

Finally, the reader should note that one of the more bizarre consequences of Theorem 1 is that we do not have to start building compositional semantics for natural language beginning with assigning of the meanings to words. We can do equally well by assigning meanings to *phonemes* or even *LETTERS*, assuring that, for any sentence, the intuitive meaning we associate with it would be a function of the meanings of the letters from which that sentence is composed. But then the cabalists had always known it.

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